Overlapping of nonlinear resonances and the problem of quantum chaos

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The motion of a nonlinearly oscillating particle under the influence of a periodic sequence of short impulses is investigated. We analyze the Schrödinger equation for the universal Hamiltonian. It is shown that the quantum criterion of overlapping of resonances is of the form $\lambda K \geqslant 1$, where K is the classical coefficient of stochasticity and λ is the functional defined with the use of Mathieu functions. The area of the maximal values of λ is determined. The idea about the emerging of quantum chaos due to the adiabatic motion along the curves of Mathieu characteristics at multiple passages through the points of branching is advanced.

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I. INTRODUCTION

The overlapping of nonlinear resonances is the criterion for the origin of dynamical stochasticity in classical Hamiltonian systems. When the conditions for this criterion have been realized it is possible to justify the transition from the dynamic Hamilton description to the statistical one and to study the behavior of the system with the help of a statistical average. Such a description is as full as possible in this range and successfully substitutes for a dynamic description, which loses its sense due to strong local instability [1,2]. However, in quantum mechanics the introduction of stochasticity is significantly difficult [3–8]. What can be considered as a quantum analog of dynamic stochasticity? What is a criterion for passing to quantum chaos? How can one quantize the system in a classical limit corresponding to the dynamic stochasticity? These are only some of the problems of quantum chaos.

In the present work an attempt is made to investigate two aspects of a general problem of quantum chaos: the criterion for the overlapping of resonances on the basis of quantum mechanics and to study the singularities of wave functions in the area, in which the classic mechanics assumes the existence of dynamic stochasticity.

Let us assume that a nonlinearly oscillating particle (Fig. 1) is under the action of a variable field,

$$H(x,p)=H_o(x,p)+H_{NL}(x)+\varepsilon V(x,t), \tag{1}$$

where

$$H_o(x)=1/2(p^2/m+\omega_o^2 m x^2),$$

$$H_{NL}=\gamma x^3+\beta x^4+\cdots,$$

$$\varepsilon V(x,t)=-(e/m)xf(t),$$

$$f(t)=f_o\Sigma(t)\cos(\omega t), \quad \varepsilon V(x,t)=\varepsilon V_o x\Sigma(t)\cos(\omega t),$$

$$\varepsilon V_o=-(e/m)f_o,$$

$$\varepsilon \ll 1. \tag{2}$$

The Hamiltonian of such type has been investigated for a long time is being with the purpose of studying a dynamic stochasticity both in classical [1] and in quantum systems [3,5,6].

Here x and p are the coordinate and the impulse of the particle, ω_o is the fundamental frequency, γ and β are the coefficients of the nonlinearity, m and e are the mass and the charge of the particle, f_o is the amplitude of the variable field, $\Sigma(t)$ is the periodic sequence of rectangular electromagnetic impulses with the duration τ and with the phase of recurring T (Figs. 2 and 3). It is supposed that $1/\omega, 1/\omega_o \ll \tau \ll T$.

The fundamental component of pumping field at frequency $\omega=\omega_o$ is able to carry out the linear resonance and cause the increase of x until the nonlinear terms proportional to x^3 and x^4 become significant in the potential (i.e., up to a neighborhood of x_L, Fig. 1). From this moment the nonlinear terms will gradually begin to detune the linear resonance (at $\omega=\omega_o$), which will reduce the resonance growth of x. Then the remaining harmonics of the pumping spectrum, concentrated in sum $\Sigma(t)$, will begin to play a role. Their role will be significant in reaching higher excitation ($x > x_L$), if criteria of the overlapping of resonances is fulfilled.

II. UNIVERSAL HAMILTONIAN: CLASSICAL CONSIDERATION

In this section we shall review the well-known results obtained in the theory of stochasticity for the nonlinearly

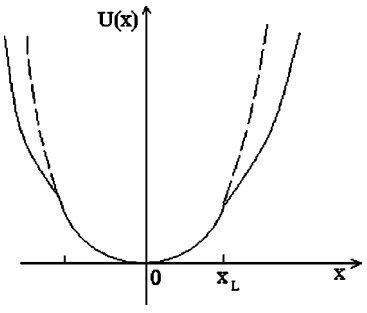

FIG. 1. Continuous line corresponds to the anharmonic potential $U_{NL}(x)=m\omega_o^2x^2/2+\gamma x^3+\beta x^4$. The dashed line corresponds to the harmonic potential $U_L(x)=m\omega_o^2x^2/2$. They coincide up to point x_L.

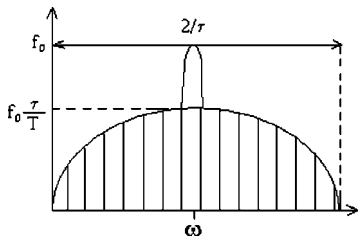

FIG. 2. Periodic series of rectangular pulses. τ is the pulse length and T is the recurring period.

oscillating classical systems. After passing in Hamiltonian (1) to the variables action-angle with the help of transformation $x=\sqrt{2I/m\omega_o}\cos\theta, p=-\sqrt{2I\omega_o m}\sin\theta$, and averaging Eq. (1) with respect to the fast phase θ, we obtain

$$H=H_o^{NL}+\varepsilon V(I)\ \ldots,$$

$$H_o^{NL}=H_o+H_{NL},$$

$$H_o=I\omega_o, H_{NL}=3\pi\beta(I/m\omega_o)^2,$$

$$\varepsilon V(x,t)=\varepsilon/2V(I)\cos(\varphi)\Sigma(t)\equiv\varepsilon V(I,\varphi,t),$$

$$V(I)=V_o\sqrt{2I/m\omega_o}, \tag{3}$$

where $\varphi=\theta-\omega t$ is the slow phase.

Let us notice that we have united H_{NL} with H_o in the unperturbed Hamiltonian H_o^{NL}. In what follows, nonlinear terms are not assumed small and the application for them of the perturbation theory is not possible. The relevant set of canonical equation looks like

$$\dot{I}=-\varepsilon\frac{\partial V(I,\varphi,t)}{\partial\phi},$$

$$\dot{\varphi}=\omega(I)+\varepsilon\partial V(I,\varphi,t)/\partial I,$$

where

$$\omega(I)=\omega_o-\omega+\omega_{NL}(I), \omega_{NL}=6\pi\beta I/(m^2\omega_o^2), \tag{4}$$

$$\varepsilon V(I,\varphi,t)=1/2\frac{\tau}{T}V(I)\cos\varphi\sum_{-1/\tau}^{1/\tau}\cos k\Omega t. \tag{5}$$

The phase φ, slow as compared to θ, remains fast in comparison with the velocity of the action I variation. The velocity of the variation of φ contains information about the nonlinear character of motion. In particular, the dependence of $\dot{\varphi}$ on I means the presence of nonlinearity in the oscillating system. Suppose that for some values of I_n the resonance between $\omega(I)$ and some component from the polychromatic pumping spectrum (5) [i.e., $\omega(I_n)\approx n\Omega$] is carried out. Then forming a slow phase $\alpha_n\approx\varphi-n\Omega t$, averaging expression (5) with respect to the fast φ, and taking into account Eq. (5), we get

$$\varepsilon V(I,\varphi,t)=1/4\frac{\tau}{T}V(I)\cos\alpha_n. \tag{6}$$

Substituting Eq. (6) in Eq. (4), we have

$$\dot{I}_n=U(I)\sin\alpha_n,$$

$$\dot{\alpha}_n=\omega(I_n)-n\Omega+dU(I)/dI\cos\alpha_n, \tag{7}$$

where

$$U(I)=1/4\frac{\tau}{T}V(I). \tag{8}$$

Equations (7) describe the nonlinear resonance. As opposed to the linear resonance at which unbounded linear growth of an amplitude is valid (in our case action I or deviation x), in the case of the nonlinear resonance (as was already mentioned), there are so-called ''phase oscillations,'' i.e., oscillations of the phase α_n and the amplitude I_n.

Let us introduce the deviation of the action, $\Delta I_n=I-I_n$, $\Delta I\ll I_n$ from the resonance value. Then it is possible to demonstrate that the Hamiltonian

$$\tilde{H}=\omega'(\Delta I)^2/2+U(I_n)\cos\alpha_n, \tag{9}$$

where $\omega'=(d\omega/dI)_{I=I_n}$, produces the set of equations of Eq. (7) type. Really, from the equilibrium conditions, $\dot{\alpha}=\dot{I}=0$, one can obtain

$$\omega(I_n)-n\Omega+\varepsilon/2dU(I_n)/dI=0,$$

$$\omega(I_n)=\omega(I_n)+\omega'\Delta I_n. \tag{10}$$

If the condition of moderate nonlinearity is just $\varepsilon\ll\mu\ll1/\varepsilon$, where

$$\mu=I_n/\omega(I_n)\left(\frac{d\omega}{dI}\right)_{I=I_n} \tag{11}$$

is the factor of nonlinearity, then with the help of Eqs. (7)–(11), we get

$$\Delta\dot{I}_n=U(I_n)\sin\alpha_n. \tag{12}$$

It is possible to obtain the equation $\dot{\alpha}_n=\omega'\Delta I_n$ for phase oscillations from set (12),

$$\ddot{\alpha}_n-\omega_{ph}^2\sin\alpha_n=0, \tag{13}$$

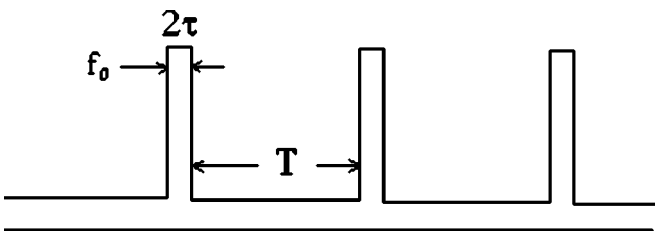

FIG. 3. Pumping spectrum $f(t)$ consists of many harmonics, multiples to the $\Omega=2\pi/T$, enveloping frequency range from ω_o+1/τ up to $\omega_o 1/\tau$.

where $\omega_{ph}=\sqrt{\omega' U}$ is the frequency of phase oscillations.

Let us notice that Hamiltonian (9) is the Hamiltonian of the mathematical pendulum, where $1/\omega'$ plays the role of mass and $U(I_n)\cos\alpha_n$ plays the role of potential energy. Taking into account that in the classical mechanics the problem of the pendulum can be solved exactly, we reduced an initial problem to the solved one.

Variation of the action with the help of Eq. (9) and (12) can be presented in the form

$$\Delta I_+ = \sqrt{(E+U)/\omega'}\, dn[\omega'\sqrt{(E+U)/\omega'}\, t;k], \quad E>U \tag{14}$$

[with the period equal to $2K(k)$],

$$\Delta I_- = \sqrt{(E+U)/\omega'}\, cn[\omega'\sqrt{(E+U)/\omega'}\, t;1/k], \quad E<U \tag{15}$$

[with the period equal to $4K(1/k)$], where E is the energy of the particle, cn and dn are Jacobian elliptic functions: the elliptic cosine and delta of amplitude; $K(k)$ is the second order complete elliptic integral, $k=\sqrt{2U/(E+U)}$ is the module of elliptic integrals. ΔI_+ and ΔI_- are deviations of the action up and below the separatrix accordingly. For $E=U$(or $k\to 1$) these two solutions are sewed together and take the form of an instanton

$$\Delta I_+ \to \Delta I_- \to \frac{\sqrt{2U/\omega'}}{c\hbar(\sqrt{2U\omega'}\,t)}. \tag{16}$$

Averaging the action deviation with respect to half period, we get the following equations:

$$\overline{\Delta I_+} = \sqrt{(E+U)/\omega'}\,\frac{1}{K(k)}\int_0^{4K(k)} dn(\tau,k)\,d\tau$$

$$= \frac{\pi\sqrt{E+U/\omega'}}{2K(k)}, \quad E>U, \tag{17}$$

$$\overline{\Delta I_-} = \sqrt{(E+U)/\omega'}\,\frac{1}{2K(1/k)}\int_{\alpha_o}^{2K(1/k)} cn(k\tau,1/k)\,d\tau$$

$$= \frac{\pi\sqrt{U/2\omega'}}{K(1/k)}, \quad E<U, \tag{18}$$

$\alpha_o=\arccos(-E/U)$.

At $E\approx U$,

$$\overline{\Delta I_-}\approx\overline{\Delta I_+}\approx\overline{\Delta I}=\pi\sqrt{U/2}\frac{1}{\ln 4\sqrt{4U/(U-E)}}. \tag{19}$$

Action variation dependence on the ratio E/U is presented in Fig. 4.

According to Fig. 4, the magnitude of ΔI_- sharply decreases as a separatrix is approached.

If during the phase oscillation ΔI_n takes enough major values (such as $\omega'\Delta I_n \geqslant \Omega$), the resonance condition $\omega(I_n)\approx n\Omega$ breaks, but other resonance condition is attuned:

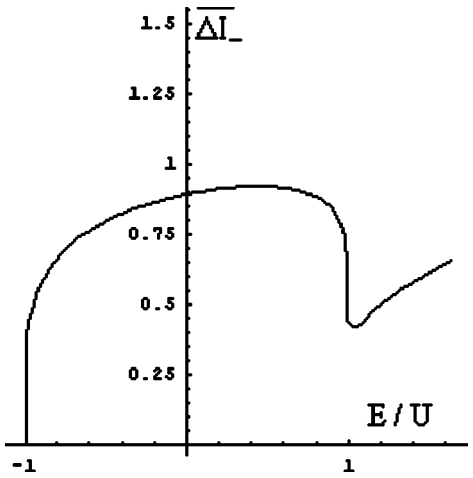

FIG. 4. $\overline{\Delta I_-}$ as a function of ratio E/U in the classical case.

$$\omega(I_n^o+\Delta I_n)\approx(n+1)\Omega. \tag{20}$$

Just at the jump to the other resonance condition there is an abruptness, which results in a stochastic wandering of spectral harmonics (6). It is the essence of overlapping of resonances, which serves as the criterion of the stochasticity emerging in the nonlinearly oscillating system. Expending $\omega(I)$ into series with respect to ΔI, and making an estimation, $\Delta I\approx\sqrt{U/\omega'}$, on the basis of Eqs. (17) and (18), it is possible to present condition (20) in the form $\sqrt{\omega' U}\approx\Omega$. Thus, in the case of overlapping of resonances the phase oscillation frequency $-\omega_{ph}$ coincides by an order of magnitude with the frequency distance between harmonics in the pumping spectrum. Usually, criterion of the dynamic stochasticity, equivalent to the overlapping of resonances, is written by introducing the stochasticity coefficient

$$K\approx\sqrt{\omega' U}/\Omega>1. \tag{21}$$

In the range of statistical motion the nonlinear oscillating system is described with the help of distribution function $\rho(t)$, for which it is possible to obtain the diffusion equation [1]

$$\frac{\partial\rho}{\partial t}=D\frac{\partial^2\rho(I,t)}{\partial I^2}, \tag{22}$$

where $D=\frac{1}{2}U^2(I)T$ is a diffusion coefficient. Now from Eq. (22) with the help of $\rho(I,t)$ it is easy to get $\overline{I}=\overline{I}_0+Dt$, where average is understood as a statistical average. Diffusion growth of the action reduces the growth of $\sqrt{\langle x^2\rangle}$ in the range of $x>x_L$. As an energy of the particle, located in a hole, is $E_o=I\omega_o$, then in the range of stochastic dynamics $\overline{E(t)}=E_o+\omega_o\sqrt{Dt}$ and ''heating'' of the particle takes place. The above-mentioned reasonings are proved by numerical calculations (Fig. 5).

The first numerical experiments for stochastic ''heating'' of a nonlinear oscillator were carried out long time ago, see Ref. [1].

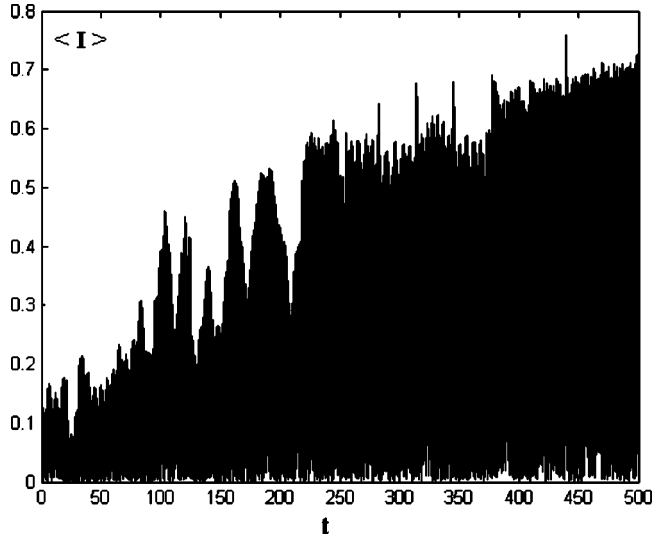

FIG. 5. Diffusion growth of the action obtained at the values of parameters $f_o=0.5, \omega_o=20, x_l=1, \Omega=0.2, T=10, \tau=1$ and the stochasticity factor $K=(\tau/T)f_o\omega' T\approx 3$.

The condition of resonance overlapping has visual interpretation on a phase plane.

The mathematical pendulum, in association with initial conditions, can make two types of motion: oscillatory and rotary. They are separated by a separatrix on the phase diagram. The overlapping of resonances on the phase plane corresponds to a touch of separatrixes (Fig. 6), if the width of the separatrix is estimated as $\omega' \Delta I_n \approx \sqrt{\omega' U}$ (in frequency units).

The dynamic stochasticity usually originates in a narrow layer near a separatrix [1,2]. Therefore, at quantum reviewing we shall be especially interested in quantum properties of the system near the separatrix. In other words, we shall be interested in a wave function of isolated nonlinear resonance in the absence of overlapping. The analysis of these properties can explain the essence of a quantum chaos.

In conclusion, we shall remark that the condition of overlapping of resonances depends on the action ΔI_n as the solution of the equations generated by the universal Hamiltonian. Therefore, at quantum reviewing, for establishing the criterion of overlapping of resonances it is enough in Eq. (20) under ΔI_n to understand the relevant magnitude obtained from the quantum equations generated by Hamiltonian (9).

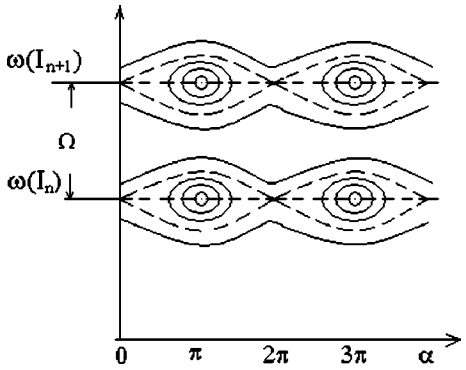

FIG. 6. Phase trajectories of the mathematical pendulum near the two resonances.

Apart from the quantum estimation for ΔI, we shall also be interested in quantum dynamics near the separatrix ($E \approx U$) as the basis of directed random motion in one-dimensional nonlinear systems.

III. QUANTUM-MECHANICAL CONSIDERATION

The quantum-mechanical consideration of isolated nonlinear resonance, as well as overlapping of two resonances, was presented in Refs. [3,5,6]. In Ref. [5] it was specified that the Schrödinger equation for a nonlinear isolated resonance can be reduced to the Mathieu's equation, and if the condition of the resonances overlapping is fulfilled the correlations drop [6] (numerical computational methods). We, in this section, are interested essentially in quantum-mechanical characteristics of the problem—wave function and energy spectrum. We shall investigate, in a quantum case, the unpredictability of hit of a system in any quantum state (analog of a stochastic stratum near a separatrix in classical mechanics).

The universal Hamiltonian (9) depends on the basic parameter of nonlinear oscillations, ω'. At quantum reviewing corresponding to the universal Hamiltonian, Schrödinger equation will also depend on ω'. So we came to the quantum-mechanical consideration of the nonlinear-oscillating system within the framework of the approximation made in Sec. I.

The Schrödinger equation relevant to Hamiltonian (9), is

$$\frac{d^2\Psi}{d\alpha^2}+\frac{2}{\chi}[E-U\cos\alpha]\Psi=0, \tag{23}$$

where $\chi=\omega'\hbar^2$.

Let us clarify the essence of the parameter χ. The value $\omega'\hbar$ is the quantum (the minimal portion) of the frequency shift stipulated by nonlinearity (i.e., the frequency quantum of nonlinearity). Hence, value $\chi=\omega'\hbar^2$ is the energy quantum of nonlinearity.

Equation (23) is the Mathieu's equation, which we shall analyze below. For now we want to get quasiclassical wave functions relevant to an approximation $\Lambda=U/\chi \gg 1$ of the Schrödinger equation. It is known that quasiclassical wave function is

$$\Psi(\alpha)=\frac{c}{\sqrt{\Delta I}}\exp\left(i/\hbar\int_0^\alpha \Delta I(\alpha)d\alpha\right), \tag{24}$$

where c is the normalizing constant and ΔI can be found from the integral of the energy,

$$\Delta I=\sqrt{2/\omega'(E+U\cos\alpha)}. \tag{25}$$

Substituting Eq. (24) in Eq. (23), after integration we get

$$\Psi_+(\alpha)=\frac{1}{2}\frac{(E+U)^{1/4}}{\sqrt{K(k)}}\frac{\exp[2i\sqrt{(E+U)/\chi}E(\alpha/2,k)]}{(E+U\cos\alpha)^{1/4}}, \tag{26}$$

$$E>U, \quad 0\leqslant\alpha\leqslant\pi; \quad \Psi_-(\alpha)=1/\sqrt{2}\frac{(U/2)^{1/4}}{\sqrt{K(1/k)}}\frac{\exp\left(i\sqrt{\frac{2}{U\chi}}[(E-U)F(\gamma,1/k)+2UE(\gamma,1/k)]\right)}{(E+U\cos\alpha)^{1/4}}, \tag{27}$$

$$0<E<U, \quad 0\leqslant\alpha\leqslant\alpha_o \quad \gamma=\arcsin\sqrt{\frac{U(1-\cos\alpha)}{U+E}}, \quad \alpha_o=\arccos(-E/U), \tag{28}$$

where $F(\cdots)$ is the elliptic integral of the first kind, $E(\cdots)$ is the elliptic integral of the second kind, and $K(\cdots)$ is the first kind complete elliptic integral.

Let us note that, as in the considered case $\Lambda\gg1$, quasiclassical wave functions (26) and (27) oscillate fast with the variation of α, having peaks at turning points $\pm\alpha_o$. It is a common property of wave functions in a quasiclassical approximation.

The wave functions corresponding to the separatrix can be obtained in the limit $E\to U$,

$$\Psi_-(\alpha)=\Psi_+(\alpha)=\Psi_s(\alpha)$$
$$=1/2\sqrt{2}\frac{1}{\ln 4\sqrt{\frac{U}{|E-U|}}}\frac{\exp(i2\sqrt{2\Lambda}\sin\alpha/2)}{(1+\cos\alpha)^{1/4}}. \tag{29}$$

According to expression (29) for a wave function near the separatrix the frequency of fast oscillations practically does not vary. The turning points $\alpha_o<\pi$ approach $\pm\pi$ and the peaks of fast oscillations become negligibly low. This is connected with the logaritmically diverging factor in Eq. (29). With the help of Eq. (29) it is easy to find an equation for nodal points of separatrix wave function, $\Psi_S(\alpha)=0$,

$$2\sqrt{2U/\chi}\sin\frac{\alpha_n}{2}=\pi/2+2\pi n, \quad n=1,2,\ldots. \tag{30}$$

Differentiating relation (24), it is possible to calculate the density of nodal points

$$\frac{dn}{d\alpha_n}=\frac{1}{2\pi}\sqrt{\frac{2U}{\chi}}\cos\frac{\alpha_n}{2}. \tag{31}$$

The high density of nodal points is provided by the major parameter of quasiclassical consideration $(2U/\chi)^{1/2}$, which is suppressed by the zeroes of the factor $\cos\alpha_n/2$ in the points $\alpha_n=\pm\pi$ (see Fig. 7). Another important characteristic of the quantum state near the separatrix is the density of energy levels. Bohr-Zommerfeld quantization condition looks like

$$I=\oint\Delta I(\alpha)d\alpha=n\hbar. \tag{32}$$

Taking into account Eq. (25), we get

$$I=\sqrt{2/\omega'}\int\sqrt{E+U\cos\alpha}d\alpha$$
$$=\frac{2}{\sqrt{U\chi}}[(E-U)K(1/k)+2UE(1/k)]=n. \tag{33}$$

With the help of Eq. (33) for the density of energy levels it is possible to obtain

$$\frac{dn}{dE}=\frac{1}{\sqrt{U\chi}}K(1/k). \tag{34}$$

In the limit $E\to U$ for the energy level density, dn/dE, we get

$$\left(\frac{dn}{dE}\right)_s=\frac{1}{\sqrt{\chi U}}\ln 4\sqrt{\frac{2U}{|E-U|}}. \tag{35}$$

As can be seen from Eq. (35), the level densities are logarithmic divergent near the separatrix.

Now let us try to evaluate the quasiclassical condition of the overlapping of resonances. For this purpose it is necessary to calculate average values ΔI for a half cycle of motion both above and below the separatrixes. Taking into account Eqs. (25)–(27), we obtain

$$\langle\Delta I\rangle^+=\int_{-\pi}^{\pi}|\Psi_+(\alpha)|^2\Delta I(\alpha)d\alpha=\pi/2\sqrt{\frac{E+U}{\chi}}\frac{\hbar}{K(k)}, \tag{36}$$

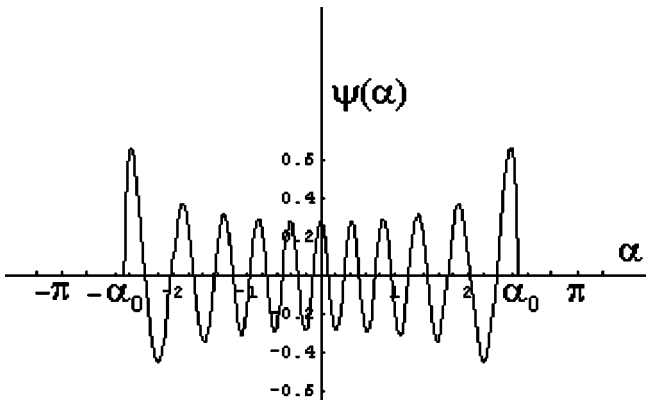

FIG. 7. Quasiclassical wave function below the separatrix, $E=0.9U, \Lambda\sim100$.

$$\langle \Delta I \rangle^- = \int_{-\alpha_o}^{\alpha_o} |\Psi_-(\alpha)|^2 \Delta I(\alpha) d\alpha = \sqrt{\frac{U}{2\chi}} \frac{\alpha_o \hbar}{K(1/k)} \tag{37}$$

and near the separatrixes, $E \approx U$,

$$\langle \Delta I \rangle^+ \approx \langle \Delta I \rangle^- \approx \langle \Delta I \rangle_S = \pi \sqrt{U/2\chi} \frac{\hbar}{\ln 4 \sqrt{\frac{2U}{|E-U|}}}, \tag{38}$$

where $\langle \cdots \rangle^{\pm}$ denotes averaging with the help of wave functions (26), (27), and (29). Let us note that

$$\langle \Delta I \rangle^- / \hbar = (U/2\chi)^{1/2} [\alpha_o / K(1/k)] \approx \Lambda^{1/2} \gg 1$$

is in good agreement with a quasiclassical condition of motion. Because of the coincidence of classical values $\overline{\Delta I}_{\pm}$ with the quasiclassical ones $\langle \Delta I \rangle^{\pm}$ it is natural that conditions of overlapping resonances will also coincide, $\omega'(1)\overline{\Delta I}_- = \omega' \langle \Delta I \rangle^- = \Omega$. Hence, one can conclude that, under quasiclassical conditions $\Lambda \gg 1$ and the condition of overlapping resonances, the stochastic ''heating'' of electron is under the conditions given in Sec. I and high excitations can be obtained (see classical case, Fig. 5). Quasiclassical expressions (36), (37) by the form coincide with the similar classical expressions (17), (18). But as opposed to the classical formulas, in quasiclassical expressions the E_n energy spectrum of quasiclassical levels should be understood under E. With the help of Eqs. (33) and (34) it is possible to determine the number of levels entrapped in a nonlinear resonance:

$$\Delta n = \frac{dn}{dE} \Delta E, \tag{39}$$

where

$$\Delta E = \hbar \omega' \langle \Delta I \rangle^-. \tag{40}$$

Δn is the important characteristics of an isolated nonlinear resonance. Using Eqs. (34), (37), (39), and (40) it is easy to show that the number of levels entrapped in nonlinear resonance is $\Delta n \simeq \alpha_o/2$. According to Eqs. (35), (38), (39), and (40) it is also easy to show that near the separatrix there is $\Delta n_s \simeq \pi/2$. While according to Eqs. (34) and (35) the density of levels increases logarithmically as the separatrix is approached, the number of levels Δn entrapped in resonance is not increased. This is caused by the sharp fall of the action variation near $U \simeq E$ (see Fig. 4). Thus the number of entrapped levels in a nonlinear resonance is not great and remains the same when approaching the separatrix. Major values Δn, as was shown in Ref. [6] by means of numerical methods, can be reached in case of overlapping resonances (Fig. 8).

Let us analyze Schrödinger equation (23). We assume that U and χ are the values of one order and that is why the quasiclassical approximation cannot be used.

In the limit case $E \gg U$ it is possible to use limiting ($U \to 0$) formulas for the Mathieu's functions [9]:

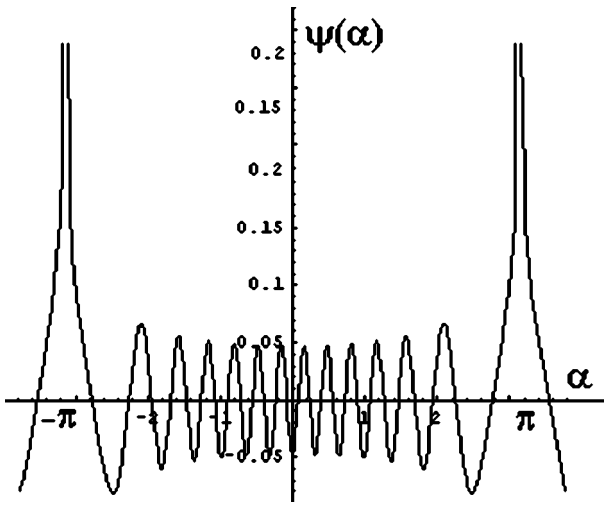

FIG. 8. Quasiclassical wave function near the separatrix, $E \sim U$.

$$\Psi_0(\alpha) = ce_o(\alpha) = 1/\sqrt{2},$$

$$\Psi_n(\alpha) = ce_n(\alpha) = \cos n\alpha,$$

$$\Psi_n(\alpha) = se_n(\alpha) = \sin n\alpha, \quad n = 1,2,3, \ldots . \tag{41}$$

These functions should satisfy the equation

$$\frac{d^2\Psi}{d\alpha^2} + \frac{2E}{\chi}\Psi = 0, \tag{42}$$

which follows from Eq. (23) at $U \to 0$. The equation describes harmonic oscillations with the frequency $\sqrt{2E/\chi}$. To reduce it in correspondence with the solution (41), it is necessary to require

$$\sqrt{\frac{2E}{\chi}} = n \quad \text{or} \quad E_n = 1/2\chi n^2, \tag{43}$$

where $n = 1,2,3, \ldots$. The last relation leads to the energy spectrum quadratically depending on the quantum number.

It is possible to use the relations obtained with the help of an averaged universal Hamiltonian for calculation of ΔI in the zero order with respect to the U/E,

$$\langle (\Delta I)^2 \rangle_n = \frac{2}{\omega'} E_n. \tag{44}$$

Then, taking into account Eq. (43), we get

$$\langle \Delta I \rangle_n \approx \sqrt{\langle (\Delta I)^2 \rangle_n} = n\hbar. \tag{45}$$

It is possible to obtain the condition of the overlapping of resonances for a frequency shift, caused by the variation of an action,

$$\delta\omega \approx \omega' \langle \Delta I \rangle_n = \omega' n\hbar \geqslant \Omega. \tag{46}$$

Using Eq. (46) the condition of resonance overlapping could be written as $\delta\omega \approx \omega' n\hbar \geqslant \Omega$. It is significantly diffi-

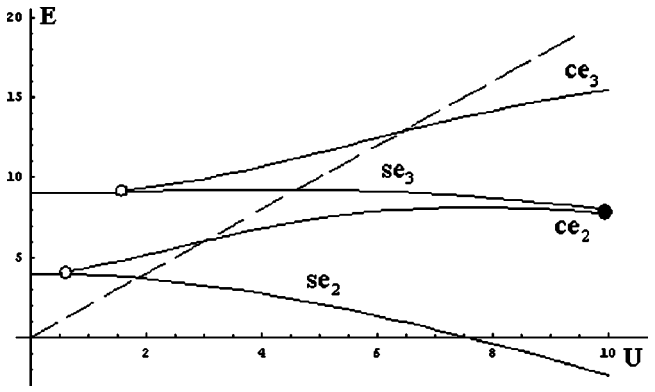

FIG. 9. Dependence of the eigenvalue E on U for different Mathieu's functions. Dashed line corresponds to the separatrix. Symbol ○ denotes branching points above the separatrix and symbol ● denotes branching points below the separatrix.

cult to fulfill this condition as compared with a similar one of quasiclassical case, because it requires excessively small Ω.

In the opposite limit case $E \ll U$ we have condition of overlapping the resonances, which is also difficult to fulfill and for this reason we do not present a detailed analysis.

It is well known that eigenvalues of the Mathieu's equation can be defined by means of the Ince-Strutt diagrams (Fig. 9) constructed for the first time to study a parametrical resonance [9]. As follows from these diagrams, each value U corresponds to a set of eigenvalues E_n and periodical wave functions $ce_n(\alpha), se_n(\alpha)$. Curves in Fig. 9 corresponding to the realized quantum states are known as Mathieu characteristics. A essential feature of Mathieu characteristics is the presence of points of branching in the neighborhood of a line $U=E$, corresponding to the classical separatrix. Moving along Mathieu characteristics from left to right in points of branching being at the left of separatrix line disappears twofold degeneration. So through the passing of separatrix line $U=E$ and reaching the point of branching, to the right of a separatrix wave functions merge again but now se_n and ce_{n-1}. The emergence of such a picture (point of branching from two sides of the separatrix) at the passage to quantum consideration is a principal characteristic describing a quantum system near classical separatrix. One can observe the appearance of unpredictability of occupied quantum levels with the help of branching points located on both sides of a separatrix. Let us suppose that one of the system parameters, for example, the amplitude of variable field U varies adiabatically [$U \to U+\epsilon\cos(\kappa t)$, $\epsilon \ll 1$, κ frequency of slow motion]. In common problems of quantum mechanics it is believed that the distance between levels in an energy distribution, depending on exterior parameters, varies synchronously with an adiabatically varying parameter, not changing in this case a quantum state. The situation is changed radically, if a quantum-mechanical problem of the definition of energy distribution and eigenfunctions is reduced to the analysis of diagrams of the Ince-Strutt type. At slow moving along the curves of Mathieu characteristics due to the adiabatic change of the amplitude of variable field, after multiple passages through the branching points it is impossible to determine exactly in which Mathieu characteristic (in which quantum state) the system can be found. One can consider this appearance as quantum analog of formation of a stochastic layer of motion in the area of classical separatrix.

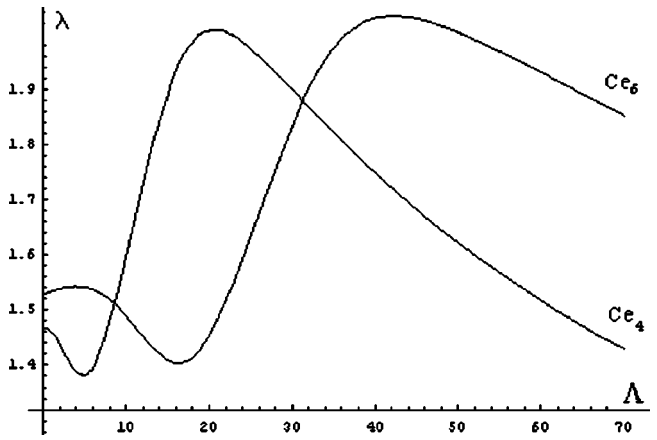

FIG. 10. Parameter λ as a function of Λ for different Mathieu's functions $ce_4 ce_6$.

Let us move to the analysis of overlapping criterion in a quantum case. Taking into account the quantum virial theorem, connecting average kinetic energy T with the average potential energy U,

$$2\langle T\rangle = \left\langle \alpha \frac{dU}{d\alpha} \right\rangle, \tag{47}$$

for variation of action ΔI with the help of Eq. (44), we obtain

$$\langle \Delta I\rangle \approx \sqrt{(\Delta I)^2)} = \lambda(\Lambda)\sqrt{\omega' U},$$

where

$$\lambda(\Lambda) = \sqrt{-\int_0^{\pi} ce_n^2(\alpha,\Lambda)\alpha \sin\alpha d\alpha} \tag{48}$$

and $\Lambda = U/\chi$. The condition of overlapping of resonances takes the form

$$\delta\omega = \omega' \langle \Delta I\rangle \approx \lambda(\Lambda)\sqrt{\omega' U} \geq \Omega \tag{49}$$

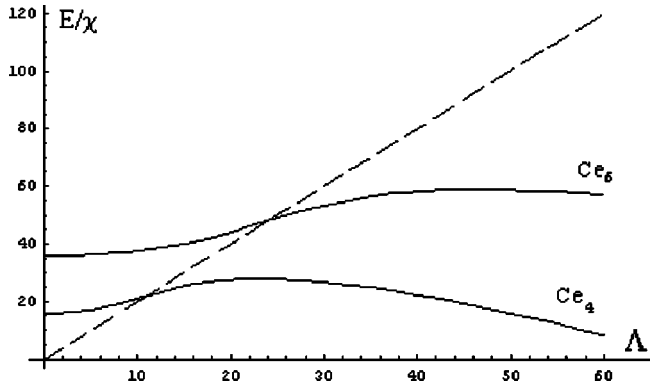

FIG. 11. Eigenvalue E as a function of U_o for Mathieu's functions $ce_4 ce_6$. Dashed line corresponds to the separatrix.

or

$$\lambda(\Lambda)K \geqslant 1. \tag{50}$$

Comparing the quantum criterion (50) with the classical (21), one can conclude that in quantum case there is an additional quantum factor $\lambda(\Lambda)$.

As can be seen from Eq. (50), the condition of resonances overlapping in the quantum case is determined by the magnitude of Λ. This condition is reduced to that how many levels of nonlinearity of energy χ can be located in U.

Thus, at quantum reviewing, the additional factor $\lambda(\Lambda)$ appears in the overlapping criterion. The physical sense of $\lambda(\Lambda)$ can be clarified. If the classic criteria of the resonances overlapping $\sqrt{\omega' U} \approx \Omega$ is fulfilled then, according to Eq. (50), the magnitude of $\lambda(\Lambda)$ determines the conditions of weakening $[\lambda(\Lambda)<1]$ or amplifying $[\lambda(\Lambda)>1]$ the resonances overlapping criterion, in comparison with the classical one. Differently, $\lambda(\Lambda)>1$ means that the condition of overlapping of resonances more easily can be fulfilled in a quantum case.

The data of numerical calculations for the magnitude $\lambda(\Lambda)$ are given in Fig. 10. As is visible from Fig. 10, $\lambda(\Lambda)>1$, i.e., the resonance overlapping criterion amplifies in the quantum case. For high values of Λ (the quasiclassical case), $\lambda(\Lambda)$ should tend to unity. In Fig. 11 the eigenvalues of an energy E/χ as functions of U_o are presented for states described by the functions $ce_4(\alpha, U)$ and $ce_6(\alpha, U)$. From Figs. 10 and 11 it is easy to see that the maximum value of $\lambda(\Lambda)$ corresponds to $(\partial E/\partial U)_n = 0$. Thus, in the quantum case the area lying below classical separatrix corresponds to the area of the maximum stochasticity.

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